Monotonic function

Problem

We are looking for a monotonous function f defined on the positive integers, with a co-dimension restricted to the interval $(0,1)$, i.e.

$$f : \mathbb{N} \to (0,1). \tag{1}$$

Furthermore, we require that for a given integer k, it holds

$$\sum_{i=0}^{k} f(i) = 1. \tag{2}$$

We also want to have a parameter α that allows us to control the steepness of the function while approaching k from 0.

Solution

We start by considering the continuous version of the problem, i.e. we are looking for a function $f = f(x;\alpha,k)$ such that

$$\int_0^k \mathrm{d}x \; f(x;\alpha,k) = 1 \tag{3}$$

where we can control the steepness through α. Consider the function

$$f(x) = \frac{\alpha+1}{k^{\alpha+1}} x^{\alpha} \tag{4}$$

with $\alpha > 0$. With only basic knowledge about calculus, one easily verifies that this f always integrates to 1, and that this f is indeed monotonous. Furthermore, we check that this f is restricted to the domain $(0,1)$ for $k > \alpha + 1$, since

$$f(x) \leqslant f(k) \overset{!}{<} 1 \tag{5a}$$
$$\Updownarrow \tag{5b}$$
$$\frac{\alpha+1}{k^{\alpha+1}} k^{\alpha} < 1 \tag{5c}$$
$$\Updownarrow \tag{5d}$$
$$k^{-1} < \frac{1}{\alpha+1} \tag{5e}$$
$$\Updownarrow \tag{5f}$$
$$k > \alpha + 1. \tag{5g}$$

Since k is in general large and α small, we know that $f(x) < 1 \; \forall x \in (0,1)$ for all practical cases. One furthermore verifies in figure 1 that α indeed controls the steepness. Finally, it is easy to check that for large k, integral (4) is a good approximation of the sum (2). For instance, for $k = 1'000$ and $\alpha = 0.5$, the sum (2) with f given by (4) evaluates to ≈ 1.00074.

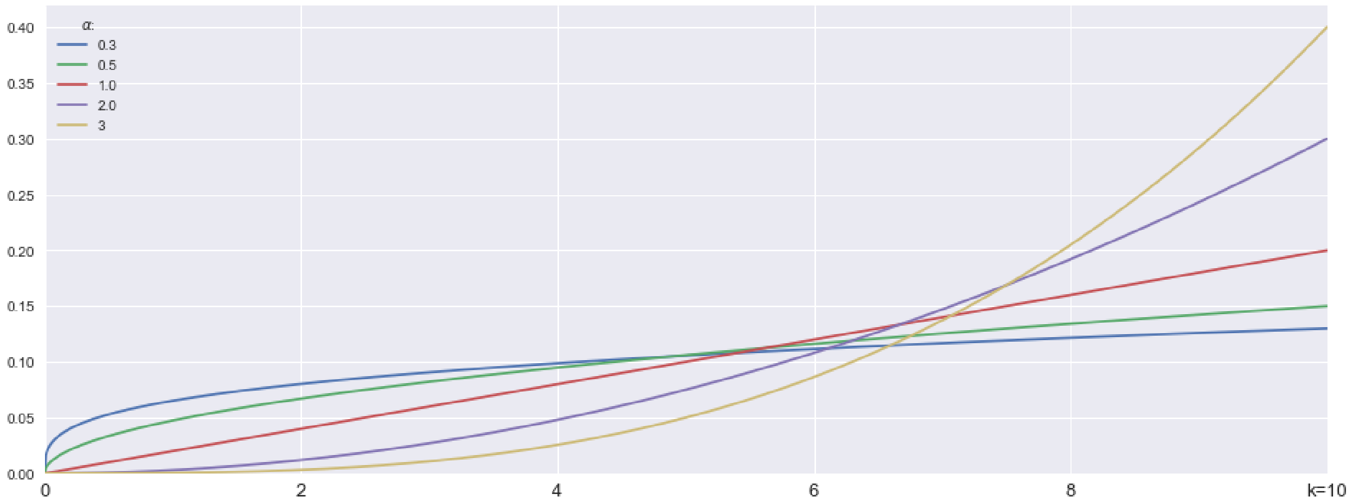

Figure 1: Visualization of the function (4) for $k = 10$ and different values of α.